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Photographic Study of Lightning

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Photographs of lightning taken with rotating film cameras near Albuquerque, New Mexico, in the summer of 1935 show three different types of electrical breakdown. In the first type the first flash starts at the cloud, proceeds about 1 km and stops abruptly in air. This was followed in 0.008 sec. by a second flash traversing the same path but extending it about 0.22 km. The third similar flash 0.01 sec. later extended the path 0.5 km. The fourth flash extended the path clear to earth 0.0098 sec. later. The second type showed the well-known "leader" type of breakdown of Schonland and others. The third type of breakdown consisted of only the main violent discharge between cloud and earth. The results are briefly discussed and a new possible mechanism for lightning breakdown pointed out.

THE phenomenon of lightning is now generally recognized as an important factor in many different geophysical problems.[1–3] For example, on the average there occur about 100 lightning flashes on the earth per second, the equivalent steady current of which amounts to about 2000 amperes.[3, 4] Also the maximum current and potential of the flash may reach 250,000 amperes and 10^9 volts, respectively.[1, 2, 3, 5] Therefore, it is clear that its influence on such phenomena as atmospherics[6–9] or "radio static," ionization in the earth's atmosphere,[2, 3] the maintenance of the earth's negative and upper air's positive charge[3] or the mechanism of the thunderstorm[1, 2] itself is important. C. T. R. Wilson has likened the thunderstorm to an immense electrostatic machine and the intercloud lightning to the sparks between its poles. Usually the positive pole of this huge machine is in the upper and the negative pole in the lower part of the cloud[10] although it is not impossible that this may be reversed in very rare cases. Also a very high voltage is usually attained between the earth and one or both poles of the cloud. This gives rise to the lightning flash between cloud and earth. In general the intercloud flashes are more abundant than those between cloud and earth. Measurements by a number of observers[1–3] agree that the average electrical field at the instant of electrical breakdown is of the order of 10,000 volts/cm instead of 30,000 volts/cm required in the case of the electrical spark.

The work reported here was undertaken with a view of securing data which, together with that already obtained by others[11–14] on various

[1] Humphreys, *Physics of the Air*, second edition, McGraw-Hill (1929).
[2] C. T. R. Wilson, Phil. Trans. Roy. Soc. **A221**, 73 (1921).
[3] Schonland, *Atmospheric Electricity* (Methuen and Co., 1932).
[4] Brookes, Meteorological Office Geophysical Memoir No. 23 (1925).
[5] Peek, J. Frank. Inst. **197**, 40 (1924).
[6] Watt and Appleton, Proc. Roy. Soc. **A103**, 84 (1923); Appleton, Watt and Herd, Proc. Roy. Soc. **A114**, 376 (1927).
[7] Norinder, J. Frank. Inst. **220**, 69 (1935); **221**, 585 (1936).
[8] Schonland, South African J. Sci. **32**, 24 (1935).
[9] Rao, Nature **136**, 683 (1935).
[10] Schonland, Proc. Roy. Soc. **A118**, 233 (1928).
[11] Schonland and Collins, Proc. Roy. Soc. **A143**, 654 (1934); Schonland, Malan and Collins, Proc. Roy. Soc. **A152**, 595 (1935).
[12] Walter, Ann. d. Physik **10**, 393 (1903); Physik. Zeits. **19**, 273 (1928).
[13] McEachron, Elect. J. **31**, 251 (1934).
[14] Boys, Nature **118**, 749 (1926); **122**, 310 (1928).

parts of the earth, should throw light on the general mechanism of lightning breakdown itself.

This first report will describe photographic observations made near Albuquerque, New Mexico, in the summer of 1935. Albuquerque has an altitude of about 5200 ft. and an abundance of night lightning particularly suited to photography because of the clear atmosphere and absence of obscuring rain, clouds, etc.

Apparatus and Results

Three cameras were used—the first with stationary photographic film, the second with film moving 80 cm/sec. and the third with film moving 850 cm/sec. The moving film cameras were of the usual type with lens stationary and film mounted on a rotating drum. For the faster camera the film was mounted on the inside of the "drum" to prevent mechanical distortion. The rotating drums were driven by hand and their speed could be held constant to better than ten percent. The lenses were achromatic and of excellent quality and the films were Eastman supersensitive panchromatic. The distance to the cloud was determined by measuring the time interval between the lightning and thunder with a stop watch and dividing this by the velocity of sound.

Fig. 1 shows a photograph taken on the film moving 80 cm/sec. in the direction indicated by the arrow F. M. The discharge was 4.5 miles away and its to talprojected two-dimensional path from cloud to earth was 1.12 miles or 1.8 km. If we follow Schonland and others in adding approximately 30 percent to this to get the length of the tortuous three-dimensional track, the total length of the discharge was about 1.5 miles or 2.4 km. It will be noted that the first flash extended only 0.98 km and ended abruptly in the air. This was followed 0.008 sec. later by a second flash traversing the same path but extending to 1.2 km and again ending abruptly in the air. The third flash occurred 0.01 sec. later and extended the path to approximately 1.7 km, while the fourth flash followed the path blazed by the others in 0.009 sec. and extended all the way to the earth. Four additional flashes (not shown in Fig. 1) traversing the same path and extending all the way to earth occurred during the next 0.08 sec.

Figs. 2 and 3 show two of a series of twelve successive flashes following the same path taken on the film moving 850 cm/sec. The two flashes shown occurred 0.017 sec. apart and were successive except for a very faint flash which occurred between them. Unfortunately the distance to these flashes was not determined with very great precision so their total length can only be given as roughly 0.8 mile or 1.2 km. It will be noted that these flashes show the well-known "leader" which starts at the cloud and proceeds to earth followed by the intense return stroke from earth to cloud.[11–13] Seven of the twelve flashes showed this "leader" and the remaining five flashes were so faint that the leader was probably too faint to be recorded. The velocities (4×10^9 cm/sec.) of the two "leaders" shown were approximately the same.

Fig. 4 shows two isolated flashes which occurred three miles away. The flashes were 0.52 mile in length and have a time separation of 8×10^{-5} sec. These flashes apparently ended the storm. On the original negative the first flash shows a possible indication of a leader which is too faint to reproduce.

Fig. 5 shows a flash taken on the film moving 850 cm/sec. It was two miles away and 1.2 miles in length as projected in two dimensions. No leader could be observed on the film although the photographic conditions were particularly favorable to record it had it existed. The original negative shows the flash repeated three times at intervals of roughly 1.2×10^{-4} sec., 1.3×10^{-4} sec. and 2.4×10^{-4} sec., respectively. This flash, together with that shown in Fig. 4, represents a type which occasionally occurs in this region. The discharges usually take place, often quite unexpectedly, after the storm has run its course and the sky is generally overcast thus causing a light rain. The thunder is cannon-like.

Discussion of Results

The three distinctly different types of the initiation of lightning shown in the above pictures strongly indicate that perhaps different mechanisms of lightning breakdown exist, depending upon the physical properties of the cloud or storm. Whatever may be the correct mechanism for the generation of high voltage by the

thunderstorm,[1, 2, 15, 16] it is known that the conditions of the path just before the lightning flash are variable. For example, the rain may be intense or slight. The wind may be high or may not exist at all. Brush discharges may or may not be present, etc. As a result, it seems logical to expect the initiation process of different flashes to be variable. Our observations illustrated in Figs. 2 and 3 confirm the important work of Schonland[11] and others[12] and is in general agreement with Schonland's conclusion that a leader precedes the main stroke in a majority of cases. Also one of our pictures, which unfortunately was not suitable for reproduction, showed the "stepped" leader recently discovered by Schonland, Malan and Collins. No attempt at this time will be made to apply the current theories[11, 15, 17, 18] of lightning breakdown to our results, but it may be of interest to note the striking resemblance of the propagation of this type of lightning discharge to the propagation of luminosity in long discharge tubes observed by one of us.[19] If an impulsive voltage is applied to one end of a long discharge tube while the other end is grounded, both the luminosity and potential wave traverse the tube from the high voltage electrode to the grounded electrode with a definite finite velocity regardless of the sign of the applied voltage.[19, 20] The velocity of the luminosity and potential wave in a discharge tube depends upon such factors as the applied voltage and pressure of gas but does not greatly differ from that of the "leader." It was pointed out[19] that this propagation of luminosity in the discharge tube could result from a "moving" space charge so that the same mechanism may also exist in the case of the lightning flash. It is also interesting to observe that the velocity of propagation is increased in both the discharge tube and in the lightning flash[11] by a weak ionization before the discharge. As the two flashes Figs. 2 and 3 are members of a number of successive strokes, the high velocity observed for the leaders is thus accounted for by the weak ionization existing before the stroke.

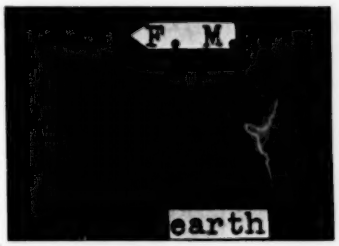

FIG. 1.

FIG. 2.

FIG. 3.

FIG. 4.

FIG. 5.

FIGS. 1, 2, 3, 4, 5. Photographs of lightning on moving film. Film motion is to the left.

The flash in Fig. 5 may have been initiated by a different mechanism as no leader was observed. Unfortunately the polarity of this flash was not obtained so that nothing is known concerning the polarity of the cloud. However, it is possible that an intercloud discharge first occurred which impressed a voltage much above the usual lightning breakdown voltage of 10,000 volts/cm between

[15] Simpson, Proc. Roy. Soc. **A114**, 376 (1927); **A111**, 56 (1926); Phil. Trans. Roy. Soc. **A209** 397, (1909).
[16] Gunn, Terr. Mag. and Atmos. Elect., March, 1935.
[17] Dorsey, J. Frank. Inst. **20**, 484 (1926).
[18] Cravatt and Loeb, Phys. Rev. **36**, 997 (1930).
[19] Beams, Phys. Rev. **36**, 997 (1930).
[20] Snoddy, Beams and Dietrich, Phys. Rev. **50**, 469 (1936).

cloud and ground. This high voltage might account for the non observance of the leader. On the other hand if Dorsey and Schonland's "electron avalanche" theory is correct, the leader would perhaps not exist when the cloud is positive with respect to ground.

The flash shown in Fig. 1 represents a type of breakdown that as far as we know has not been previously recorded. The times between the first three flashes which end in the air are too long to make it possible to associate them with leaders. Also their intensity and appearance indicates that each is a type of "main" stroke. As this picture was made with the slow camera (80 cm/sec. film speed), the leader could probably not have been clearly resolved if it existed. The fact that the first three flashes did not reach the earth suggests that the voltage was interrupted before the flash reached the earth. Most charged clouds extend over a wide space (order of kilometers[2]). Therefore it is possible that the resistance within the cloud was so great that energy could not be fed into that part of the cloud where the flash was initiated fast enough to keep the flash going. On this hypothesis the flash would stop until the potential built up and would then discharge again just as observed. When the flash finally reached the earth the high voltage no longer existed across the length of the flash because of the comparatively low resistance of the path but was applied between the parts of the cloud. Consequently either spark or glow discharges were probably formed within the cloud. These immediately lowered its internal resistance and permitted the violent fourth flash. The existence of these spark and glow discharges may account for the fogging of the original film near the beginning of the flash.

We wish to point out a mechanism of lightning breakdown which is able to account for many of our results. It is based upon the principle of cascaded spark gaps as used in the spark gap transmission line developed at the University of Virginia. This consists of a number of spark gaps connected in series in the form of a transmission line with one end grounded. When an impulsive voltage more than necessary to break down the first gap but not great enough to break down all of them is applied to the other end, the first gap breaks down followed by all the others in succession. In the case of the lightning flash the raindrops, groups of ions or other conducting regions in the air act as the electrodes for the "spark gap line."

The impulsive potential could easily be applied by a local spark or discharge in or close to the cloud since it is easy to see how local fields near the cloud may reach 30,000 volts/cm when the average field is 10,000 volts/cm. According to Humphreys[1] there is good reason for believing that conducting regions considerably larger than the raindrop may frequently exist in a thunderstorm. Also Zeleny[21] and others[22, 23] have shown that a drop of water in a field of 10,000 volts may produce a brush discharge and thus make fairly large regions of the air conducting.

In the actual case the charges on these conducting regions as well as their capacities to surrounding regions would of course be variable, but for the purpose of a simple illustration let us assume the capacities equal and the charges zero. If the capacity of each conducting region to the other similar conducting regions in the direction of the field is C_1 and its capacity to those perpendicular to the field is C_2 and if the impulsive potential applied by a spark in the cloud is V, it can easily be shown that the potential V_1 between the first two similar conducting regions nearest the cloud is given by

$$V_1 = V\left[\frac{C_2/2+(C_2^2/4+C_1C_2)^{\frac{1}{2}}}{C_1+C_2/2+(C_2^2/4+C_1C_2)^{\frac{1}{2}}}\right]$$

and V_k between the K and $(K+1)$st similar conducting regions in the direction of the field is

$$V_k = V\left[\frac{C_2/2+(C_2^2/4+C_1C_2)^{\frac{1}{2}}}{C_1+C_2/2+(C_2^2/4+C_1C_2)^{\frac{1}{2}}}\right]^k.$$

If the conducting regions are randomly spaced (as would be approximately true if they were raindrops themselves) on the average C_2 should be roughly twice C_1, but assuming them equal for the purpose of a rough calculation $V_1 = 0.6V$ while $V_k = (0.6)^k V$. In other words, most of V would be applied between the first two conducting regions until the gap separating them broke

[21] Zeleny, Phys. Rev. **3**, 69 (1914); **16**, 108 (1920); J. Frank. Inst. **219**, 659 (1935).
[22] Macky, Proc. Roy. Soc. **A133**, 565 (1931).
[23] Tonks, J. Frank. Inst. **221**, 613 (1936).

down. This would apply the potential V_1 between the second and third conducting regions until the gap separating them broke down, etc. As a result the lightning flash could take place when the voltage of the cloud was much less (easily at an average field of 10,000 volts/cm) than the normal sparking voltage. Also the luminosity would normally start at the cloud and move toward the ground. In fact if the conducting regions are large, the so-called time lag of the spark gap breakdown should hold up the discharge momentarily at each conducting region, which would naturally account for the stepped leaders discovered by Schonland[11] and observed in our work.

This investigation is being continued during the summer of 1936 with improved apparatus with the hope of collecting more precise and a wider range of data.

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Concentration Fields and Mechanical Forces in Metabolizing Systems

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Some diffusion problems arising in biophysics are discussed for simple two-body cases. The steady state concentration field is determined, by a method of images, for cases of constant inflow over the surfaces of two external spheres, or two long parallel cylinders; both when the bodies are in an inactive environment, and when they are aligned with an external gradient. As an application, there is calculated the resultant force on each body arising from a normal surface pressure proportional to the concentration. The numerical value of these forces is of the same order of magnitude as that approximately estimated previously by N. Rashevsky.[3]

BIOLOGY affords numerous examples of systems in which there is continual diffusion of dissolved substances to and from bodies which are seats of continual chemical reaction; and there arises the problem of determining the concentrations in different parts of the system. According to the kinetic theory of solutions, there will be a normal pressure on the surfaces

$$p = Kc, \tag{1}$$

where c = concentration, and K is approximately RT/M, the osmotic pressure constant for the diffusing substance. Accordingly, variation in concentration over their surfaces will give rise to variations in this pressure, and thus to forces, both resultant and distorting, on the bodies themselves. Certain cases of single near-spherical bodies have been treated by Rashevsky[1-3] and Reiner[4] in discussing the stability of spherical cells; we here consider simple cases involving two bodies.

The general problem is difficult, but it is possible to handle certain cases in comparatively simple manner. To this end we suppose

(I) The substance transport is effected entirely by diffusion in a stationary external medium.
(II) The diffusion coefficient D of the medium is constant throughout.
(III) The metabolizing bodies have been at rest, and their reaction rates constant, for a long enough time that we may consider the asymptotic steady state of the diffusion process to have been reached.
(IV) There is no consumption or production in the region outside the specific bodies being considered.

(I) neglects the complications of convection currents in the surrounding medium, and of force fields acting on the diffusing substances, leaving for the flow $v = -D \operatorname{grad} c$; (II) reduces the diffusion-continuity equation: $-\operatorname{div} v = Q + \partial c/\partial t$; to $D\Delta^2 c = Q + \partial c/\partial t$; (III) removes the time variation term; and (IV) the Q term. Thus, under these conditions, $\Delta^2 c = 0$; i.e., the concentration is simply a harmonic function outside the active bodies. To complete its specification boundary conditions are assigned. This will be done for various cases, being taken in the form $\partial c/\partial n$ = constant over each body surface. The more general form $\partial c/\partial n = kc + b$ would be desirable, but has not yet been treated.

As a particular utilization of the solutions obtained, we shall evaluate the vector integral

$$F = -K \int c\, dS \tag{2}$$

over the surface of each body. This gives the resultant force on the body due to any pressure of the form (1).

Two spheres

For two spheres, external to each other, and a given value c_0 of the concentration at large distances, the conditions to be satisfied are

$$\begin{aligned} &\Delta^2 c = 0 \text{ outside the spheres,} \\ &\partial c/\partial n = k_1 \text{ on sphere No. 1;} \quad = k_2 \text{ on No. 2,} \\ &c = c_0 \text{ at } \infty. \end{aligned} \tag{3}$$

Since c is harmonic, it may be thought of as a potential, and the problem approached by trying to find a distribution giving rise to it. (3) is equivalent to

$$\begin{aligned} &c = c_0 + c_1 + c_2, \\ &\Delta^2 c_1 = \Delta^2 c_2 = 0, \\ &\partial c_1/\partial n = k_1 \text{ on No. 1;} \quad = 0 \text{ on No. 2,} \\ &\partial c_2/\partial n = 0 \text{ on No. 1;} \quad = k_2 \text{ on No. 2,} \\ &c_1 = c_2 = 0 \text{ at } \infty. \end{aligned} \tag{4}$$

[1] N. Rashevsky, Physics **5**, 374 (1934).
[2] N. Rashevsky, Cold Spring Harbor Symposia on Quantitative Biology **2**, 188 (1934).
[3] N. Rashevsky, Physics **6**, 349 (1935).
[4] Reiner, Physics **7**, 368 (1936).

Suppose we attempt to determine c_1. Its boundary condition on the first sphere may be satisfied by placing a point source M at the center, with potential $c=M/r$, $\partial c/\partial r=-M/r^2$; and taking $M=-k_1r_1^2$. However this does not give zero normal derivative on No. 2, and to remedy this the appropriate *image* is introduced inside the latter; namely a distribution whose potential combined with that of the first source does have zero derivative on No. 2. This component in turn has non-zero derivative on No. 1, hence another image is introduced there, etc.; the successive images becoming weaker each time. The second and last of conditions (4) are always satisfied in this process, and the sequence of images gives a series which converges to a function satisfying the third as well.

Now,[5] the image in a sphere of an external point source M, d distant from the center, is a source Mr/d at the inverse point, plus a uniform line sink of total strength $-Mr/d$ thence to the center. The image back in the first sphere of such a point with a sink tail is a similar arrangement, with the sink stretched between the inverse points of the ends of the first sink; etc.

Just as c is a sum of components from the various images, so may the total force on either sphere be thought of as the sum of components, each arising from a single image. Expressions for these will now be obtained.

The integral (2) over a simple closed surface is equal to the volume integral throughout the interior

$$-K\int \text{grad } c\cdot dv.$$

But this is merely the force on a body of uniform density "$-K$" in a gravitational field of potential (strictly, negative potential or force function) c. Hence for a point source of strength (mass) M outside the sphere, the resulting force is as if all the material of the sphere were collected at the center,

$$F=KMV/d^2. \tag{5}$$

If the source be inside, it is as if all the mass inside the concentric sphere on which it lies were collected at the center, and all that of the spherical shell outside disregarded;

$$F=KMV(d/r)^3/d^2=KMVd/r^3. \tag{6}$$

Here d is the distance of the point from the center, V is the sphere volume, and the direction of the forces is from the center of the sphere away from the point; the negative density giving repulsion instead of attraction.

Uniform line source

This is taken as collinear with the center of the sphere, lying entirely on one side of it, and entirely inside or outside the sphere; these being the only cases here needed.

(a) *Outside.* If total strength is M, and the ends are d_1 and d_2 from the sphere center; the total force directed away from the source is, using (5)

$$F=MKV/(d_2-d_1)\int_{d_1}^{d_2} dx/x^2=KVM/d_1d_2. \tag{7}$$

(b) *Inside.* Employing (6)

$$F=KVM/r^3(d_2-d_1)\int_{d_1}^{d_2} xdx$$
$$=KVM(d_2+d_1)/2r^3. \tag{8}$$

Complete image

This comprises a point source M at one end of a line sink of equal and opposite strength. If outside the sphere, M is pointed towards the center; if inside, it is away from the center.

(a) *Outside.* By (5) and (7) the net force away from the image is

$$F=KVM[1/d^2-1/d(d+\Delta)]$$
$$=KVM\Delta/d^2(d+\Delta), \tag{9}$$

where d is the distance of M from the center, and Δ the length of the sink.

(b) *Inside.* Using (6) and (8)

$$F=KVM[d-(d+d+\Delta)/2]/r^3$$
$$=KVM\Delta/2r^3 \tag{10}$$

directed away from the image.

Note that in both cases the force is proportional to $M\Delta$, which is a sort of "moment" of the source-sink combination.

Consider now the actual steps in the process of building up the function c_1.

[5] A. B. Bassett, *Treatise on Hydrodynamics* (Cambridge), Vol. I, Chap. III.

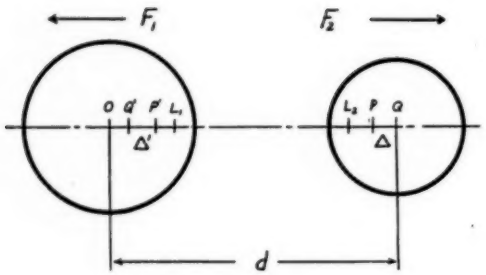

FIG. 1.

(I) Point source M at center of No. 1. This contributes nothing to the force on No. 1, $F_{11}=0$. To the force on No. 2 it supplies, by (5), a term $F_{21}=KV_2M/d^2$.

(II) Image in No. 2. This has strength, $M'=Mr_2/d$ and spread $\Delta=r_2^2/d$. The force on No. 1 is, by (9),

$$F_{12}=KV_1M'\Delta/d(d-\Delta)^2=KV_1Mr_2^3/d^3(d-\Delta)^2,$$

while (10) gives

$$F_{22}=KV_2M'\Delta/2r_2^3=KV_2M/2d^2.$$

(III) Image in No. 1. This has strength

$$M''=M'r_1/(d-\Delta)=Mr_1r_2/d(d-\Delta)$$

and spread

$$\Delta'=r_1^2/(d-\Delta)-r_1^2/d=r_1^2r_2^2/d^2(d-\Delta)$$

giving, by (10),

$$F_{13}=KV_1M''\Delta'/2r_1^3=KV_1Mr_2^3/2d^3(d-\Delta)^2$$

and by (9), $F_{23}=KV_2M''\Delta'/f^2g$ where

$$f=\overline{QP'}=d-\overline{OP'}=d-r_1^2/(d-\Delta)$$
$$=(d^2-r_1^2-r_2^2)/(d-\Delta),$$

$$g=\overline{QQ'}=d-\overline{OQ'}=d-r_1^2/d=(d^2-r_1^2)/d$$

and hence

$$F_{23}=KV_2Mr_1^3r_2^3/d^2(d^2-r_1^2)(d^2-r_1^2-r_2^2)^2.$$

This procedure may be continued in perfectly straightforward manner if more terms be desired, as would be the case for the spheres close together.

We can divide the force on either sphere into 2 series: one arising from sources inside itself, and the other from sources in the other sphere. Thus

$$F_1=(0+F_{13}+\cdots)+(F_{12}+\cdots),$$
$$F_2=(F_{22}+\cdots)+(F_{21}+F_{23}+\cdots). \tag{11}$$

Now in passing from an image in one sphere to its succeeding image in the other, the moment is reduced, $M'\Delta'\leqq M\Delta r^3/d^3$ and by the time we have gotten back to the first sphere

$$M''\Delta''=M\Delta r_1^3r_2^3/h^6\equiv M\Delta\varphi,$$

where h^2 is some value between d^2 and $L^2\equiv\overline{OL_2}\cdot\overline{QL_1}\equiv l_1l_2$. For the successive images crowd in toward the two limiting points L_1 and L_2 of the system, remaining always outside them (i.e., outside the segment $\overline{L_2L_1}$); and yet always between the centers O, Q. This gives the approximate ratio of successive terms in each of the parentheses above. But

$$1+\varphi+\varphi^2+\cdots=1/(1-\varphi);$$

hence (11) gives

$$F_1=(F_{13}+F_{12})/(1-\varphi),$$
$$F_2=F_{21}+(F_{22}+F_{23})/(1-\varphi).$$

It will be evident how to find upper and lower bounds for the forces. Closer bounds may be gotten by calculating more terms in (11) before the series are summed by dividing the last terms by $(1-\varphi)$. This not only gets one out to smaller terms before the approximating begins, but permits closer specification of h because by now the images have drawn in farther from the centers; i.e., $L\leq h\leq d'<d$.

As an approximation we take

$$F_1=(F_{13}+F_{12})/(1-\varphi)=MRV_1(r_2/d)^3,$$
$$F_2=(F_{21}+F_{22})/(1-\varphi)=MRV_2, \tag{12}$$

where $R\equiv 3K/2d^2(1-\varphi)$, where now upper and lower bounds are obtained by letting d (and h) range between d and $l\equiv$least of l_1, l_2.

The above is for c_1 only. A similar result holds for c_2, and thus for $c=c_0+c_1+c_2$ the total separating forces corresponding to (12) are

$$F_1=RV_1[M_2+M_1(r_2/d)^3],$$
$$F_2=RV_2[M_1+M_2(r_1/d)^3], \tag{13}$$

where $M_i=-k_ir_i^2$; or, since the total sphere production rates are $P_i=-(\text{Area})D\cdot\partial c/\partial n=-4\pi r_i^2k_iD$, $M_i=P_i/4\pi D$.

As an idea of the order of approximation involved, the maximum error in the first terms of (13) for various separations of equal sized spheres is as follows: For this case $2L=2l=d+(d^2-4r^2)^{\frac{1}{2}}$.

d/r	Percent error
3	32
4	15
5	9
6	6
10	2

The above still gives only the forces connected with variation in concentration of a single substance produced or destroyed, and since for a strictly steady state there can be piling up or withdrawal of mass from the bodies, it is necessary to consider at least two substances. However, if the rate of production or consumption of the substance is small as compared with the total mass of the metabolizing system, we may get a reasonable approximation to steady state conditions. This limitation has been explicitly stated by Rashevsky in his previous studies of similar problems,[6, 7] and has been shown to apply to all biological cases. In any case the extension to several substances is readily effected, for the forces merely add. P_i, D, and K may vary with the substance, but the result is again of the form (13), which may be taken as representing the complete solution for spheres considerably separated.

It is to be noted that the forces are by no means equal and opposite. We are not here dealing with central forces; and not merely the spheres, but the entire system, is concerned in the force balance. For the spheres, the forces may well be in the same direction, thus tending to pursuit phenomena; though of course we have not considered the dynamical case of actual motion.

Sphere in external gradient

In the preceding the spheres have been in an inert environment; i.e., if they stopped metabolizing, or were removed, the concentration would assume a constant value throughout the region. We consider now a slightly more general problem. Suppose a sphere introduced into an infinite region in which there was previously a uniform concentration gradient $c_0=gx+b$. The new concentration is to satisfy

$$\Delta^2 c=0, \quad \partial c/\partial n=k \text{ on the sphere surface,}$$
$$c\rightarrow c_0 \text{ at } \infty.$$

We may take $c=c_0+c_1+c_2$ where

$$\begin{aligned} \Delta^2 c_1=\Delta^2 c_2&=0,\\ \partial c_2/\partial n&=k \text{ on sphere},\\ \partial(c_0+c_1)/\partial n&=0 \text{ on sphere},\\ c_1=c_2&=0 \text{ at } \infty. \end{aligned} \tag{14}$$

I.e., c_1 is to be the image in the sphere of the original field, and c_2 derives simply from a point source at the center. Now $\partial c_0/\partial n=g\cos\varphi$, where $\varphi=$angle measured from the positive x axis. Hence $\partial c_1/\partial n=-g\cos\varphi$. The potential of a dipole at the sphere center, with its source end pointed along the positive x axis is $c=M/r^2\cos\varphi$, from which $\partial c/\partial n=\pi 2M/r^3\cos\varphi$, which gives c_1 for $M=ga^3/2$. c_2 is given by P/r, requiring $-P/a^2=k$, or $P=-ka^2$. ($a=$sphere radius.)

Thus $\quad c=gx+b+ga^3\cos\varphi/2r^2-ka^2/r.$

The force due to c_0 term is $-K\int \text{grad } c_0\cdot dv=KVg$ to the left. Thinking of the dipole as formed by equal and opposite sources coalescing in such a way that $m\Delta$ remains constant, we have from (6), remembering that M here plays the role of Md there, that the force due to the c_1 term is

$$KVM/a^3=KVg/2. \tag{15}$$

c_2 contributes nothing. Thus the total force is

$$F=3/2KVg. \tag{16}$$

Two spheres in external field

The conditions on c are now

$$\begin{aligned} &\Delta^2 c=0,\\ &\partial c/\partial n=k_1 \text{ on No. 1}; \quad k_2 \text{ on No. 2},\\ &c\rightarrow c_0=gx+b \text{ at } \infty, \end{aligned}$$

which are equivalent to

$$\begin{aligned} &c=c_0+c_1+c_2+c_3+c_4; \quad \Delta^2 c_1\cdots c_4=0;\\ &\partial c_1/\partial n=-g\cos\varphi \text{ on No. 1}; =0 \text{ on No. 2};\\ &\partial c_2/\partial n=0 \quad \text{on No. 1}; =-g\cos\varphi \text{ on No. 2};\\ &\partial c_3/\partial n=k_1 \quad \text{on No. 1}; =0 \quad \text{on No. 2};\\ &\partial c_4/\partial n=0 \quad \text{on No. 1}; =k_2 \quad \text{on No. 2};\\ &c_1\cdots c_4=0 \text{ at } \infty. \end{aligned}$$

[6] N. Rashevsky, Physics **1**, 143 (1931).
[7] N. Rashevsky, Protoplasma **14**, 99 (1931).

c_3 and c_4 are the functions previously determined for the bodies in an inert environment. c_1 is obtained by placing at the center of No. 1 such a dipole as cancels the normal derivative of c_0 at the surface, and following out the appropriate image sequence to give zero derivative for c_1 on the other sphere without disturbing that on the first. Similarly for c_2.

If the external gradient is parallel to the line of centers, so are the dipole axes; and in this case the successive images are readily handled. The image of a dipole M, d distant from a sphere of radius r, is one at the inverse point with strength $-(r/d)^3M$; i.e., it points the other way from the first.[5]

The force on a sphere due to such a dipole inside has been found in (15). For a dipole outside, (5) gives

$$KV[m/d^2-m/(d+\Delta)^2]=KVm\frac{d^2+2d\Delta+\Delta^2-d^2}{d^2(d+\Delta)^2},$$

which as $\Delta\to 0$ becomes $2KVm\Delta/d^3$. Replacing $m\Delta$ by M, we have $2KVM/d^3$. The successive strength ratios in the same sphere are $r_1^3r_2^3/d^6=\varphi$ as before. Hence, for c_1, we have

$$\begin{aligned}F_1&=(KV_1M/r_1^3)(1+\varphi+\varphi^2+\cdots)\\&\quad+(2KV_1Mr_2^3/d^6)(1+\varphi+\cdots)\\F_2&=(2KV_2M/d^3)(1+\varphi+\cdots)\\&\quad+(KV_2Mr_2^3/r_2^3d^3)(1+\varphi+\cdots)\end{aligned}$$

or
$$\begin{aligned}F_1(1-\varphi)&=KV_1g/2+KV_1g\varphi,\\F_2(1-\varphi)&=3KV_2g(r_1/d)^3/2;\end{aligned}$$

while for c_2, $F_1(1-\varphi)=-3KV_1g(r_2/d)^3/2$,
$F_2(1-\varphi)=-KV_2g/2-KV_2g\varphi$.

The c_0 terms are (the external field gradient is to the right in the diagram)

$$F_1=KV_1g,\quad F_2=-KV_2g,$$

and the terms due to c_3 and c_4 are those previously calculated in (13). By neglecting φ, the forces independent of c_3 and c_4 are

$$\begin{aligned}F_1&=3KV_1g[1-(r_2/d)^3]/2,\\F_2&=-3KV_2g[1-(r_1/d)^3]/2.\end{aligned}\qquad(17)$$

It is to be noted that in each of (13), (16), (17), the result is approximately 3/2 of what would be obtained by simple superposition of the fields for each sphere in the absence of the other; i.e., the effect of making the normal derivative constant over the sphere surface is to increase by 1/2 the force calculated for a "transparent" sphere which does not disturb the field. Therefore, using biologically plausible values for the various constants involved, we shall find by this exact treatment the same order of magnitude for the forces as that calculated by Rashevsky[3] on the basis of much rougher approximation.

Parallel cylinders

If the cylinders are long enough that we may neglect variation in the direction of their axes in regions far removed from their ends—the usual infinite cylinder assumption—the problem reduces to two dimensions and may be handled in manner analogous to the preceding. There is one difference, however; in this case the potentials are logarithmic and do not satisfy the last condition of (4). Accordingly, a slightly different formulation is necessary. Describe a cylindrical surface S of radius A about some axis near and parallel to those of the cylinders under discussion. Then as A is increased the potential on S due to sources inside the cylinders tends to $-M\log A$, where $M=$sum of source strengths. Hence, instead of (4), we state the problem

$$\begin{aligned}c&=c_0+c_1+c_2+c_3;\\\Delta^2c_1&=\Delta^2c_2=0;\\\partial c_1/\partial n&=k \text{ on No. 1};\quad =0 \text{ on No. 2};\\\partial c_2/\partial n&=0 \text{ on No. 1};\quad =k_2 \text{ on No. 2};\\c_3&=+M\log A\end{aligned}\qquad(18)$$

for a large value of A. Thus c reduces sensibly to a constant c_0 on S. So to speak, we have taken a particular shape (cylindrical) for the surface at ∞. If $M=0$; i.e., if one cylinder consumes what the other produces, this restriction may be ignored.

The rest of the procedure is as before. The image of a point source M outside a cylinder ("point" because we are working in a normal cross section; actually it is a line parallel to the cylinder) is a pair of points: $+M$ at the inverse point, and $-M$ at the center.[5] The image of such a pair in another cylinder is another pair at the respective inverse points, those at the center canceling out; etc.

The forces per unit length of cylinder due to various combinations are as follows:

Point source

The force on a cylinder of uniform density "$-K$" due to a uniform parallel line, of mass M per unit length, outside the cylinder is as if the mass of the cylinder were collected uniformly along its axis,

$$F=KMA/d; \tag{19}$$

while if the line be inside, only as much of the cylinder as is nearer the axis is considered,

$$F=KMA(d/r)^2/d=KMAd/r^2. \tag{20}$$

A is the cylinder cross-section area, and the forces are away from the source.

Complete image

This consists of a pair of points collinear with the center of cylinder section, and with equal and opposite strengths. If outside, the $+M$ point is toward the center; if inside, $-M$ point is toward the center.

(a) *Outside.* By (19)

$$F=KMA[1/d-1/(d+\Delta)] =KAM\Delta/d(d+\Delta), \tag{21}$$

where d = distance of $+M$ from the center.

(b) *Inside.* From (20)

$$F=KAM\Delta/r^2. \tag{22}$$

Proceeding with the image sequence for c_1:

(I) Source at center of No. 1.

$$F_{11}=0,$$
$$F_{21}=KA_2M/d.$$

(II) Image in No. 2. $+M$ at the inverse point; $-M$ at the center. By (21)

$$F_{12}=KA_1M\Delta/d(d-\Delta)=KA_1Mr_2^2/d^2(d-\Delta)$$

and (22)

$$F_{22}=KMA_2\Delta/r_2^2=KA_2M/d.$$

(III) Image in No. 1.

$$\Delta'=r_1^2[1/(d-\Delta)-1/d]=r_1^2\Delta/d(d-\Delta).$$

Hence

$$F_{13}=KA_1M\Delta'/r_1^2=KMA_1\Delta/d(d-\Delta)=F_{12},$$

$$F_{23}=KMA_2r_1^2r_1^2/d(d^2-r_1^2)(d^2-r_1^2-r_2^2).$$

In passing from an image to the next one in the same cylinder, M remains the same, but Δ is reduced

$$\Delta''=\Delta r_1^2r_2^2/h^4\equiv\varphi\Delta,$$

where, as before, $L\leq h\leq d$. Hence corresponding to (12)

$$\begin{aligned} F_1&=(F_{13}+F_{12})/(1-\varphi)=2F_{12}/(1-\varphi) \\ &=RA_1M(r_2/d)^2, \\ F_2&=F_{21}+(F_{22}+F_{23})/(1-\varphi)\leqq 2F_{21}/(1-\varphi) \\ &=RA_2M, \end{aligned} \tag{23}$$

where $R\equiv 2K/d(1-\varphi)$.

This is for the activity of the first cylinder only; for both

$$\begin{aligned} F_1&=RA_1[M_2+(r_2/d)^2M_1], \\ F_2&=RA_2[M_1+(r_1/d)^2M_2], \end{aligned} \tag{24}$$

where, since the component of the first source alone is

$$c=-M\log r; \qquad \partial c/\partial n=-M/r,$$

we have $M_i=-r_ik_i$.

Or, in terms of total production per unit length, $P_i=-2\pi r_ik_iD$,

$$M_i=P_i/2\pi D.$$

Cylinder in external gradient

The axis is taken perpendicular to the direction of the previous field. c_1 (cf. [14]) now derives from a logarithmic doublet

$$c=M\cos\varphi/r; \qquad \partial c/\partial n=-M\cos\varphi/a^2,$$

giving $M=ga^2$.

Also $c_2=P\log r; \qquad \partial c_2/\partial n=P/a$

and hence $P=ka$. Thus (apart from the term c_3 of [18]

$$c=gx+b+ga^2\cos\varphi/r+ka\log r$$

with force per unit length of cylinder, using (22),

$$F=KgA+KAM/a^2=2KgA \tag{25}$$

directed along the negative x axis.

Two parallel cylinders in external gradient

The axes are taken perpendicular to the field, but their plane parallel to the field. In this case

we have all doublets along line of centers in the cross-section diagram. The image of a doublet M outside a circle and directed toward its center is an oppositely directed one of strength $-(r/d)^2$ at the inverse point.

The force per unit length due to interior doublet was found in (25) to be KAM/r^2 toward the negative doublet axis. If it be outside the circle, (21) gives in the limit KAM/d^2 in direction of the positive doublet axis. Hence, as in work preceding (17), we have from the c_1 component

$$F_1=(KA_1M/r_1^2)(1+\varphi+\cdots)+(KA_1Mr_2^2/d^4)(1+\varphi+\cdots),$$
$$F_2=(KA_2M/d^2)(1+\varphi+\cdots)+(KA_2Mr_2^2/r_2^2d^2)(1+\varphi+\cdots)$$

or

$$F_1(1-\varphi)=KA_1g+KA_1g\varphi,$$
$$F_2(1-\varphi)=2KA_2g(r_1/d)^2;$$

and from C_2

$$F_1(1-\varphi)=-2KA_1g(r_2/d)^2,$$
$$F_2(1-\varphi)=-KA_2g-KA_2g\varphi.$$

The c_0 terms are

$$F_1=KA_1g, \qquad F_2=-KA_2g.$$

Hence the field-dipole forces are, corresponding to (17),

$$F_1=2KA_1g[1-(r_2/d)^2],$$
$$F_2=-2KA_2g[1-(r_1/d)^2]. \tag{26}$$

From (23), (25), (26) it is seen that the factor by which the boundary conditions multiply the force on cylinders is 2, as against 3/2 for spheres.

As far as concerns the shapes of the bodies here treated, they might be thought of in the one case as spherical cells, inert spherical particles (special case of zero reaction rate), or spherical drops dissolving or condensing; and in the other as long cylindrical cells, say nerve fibers, or perhaps blood vessels, bones of the right shape (inert cylinders), and similar structures.

However, some of the other conditions may not be appropriate for any particular application, and any case selected requires individual scrutiny. Thus, for dissolving drops or food particles the supposition that a steady state has been reached may or may not be allowable; in case of cells with volume reactions distributed through the interior it is necessary to consider another distribution function for the internal region[1-3] and the boundary conditions matching this with the external concentration at the cell surface take more complicated form than those used above. In fact since the $D\cdot\partial c/\partial n$ measures the rate at which substance is being carried to unit surface area, $\partial c/\partial n=k$ describes a constant rate surface reaction and hence would not in general be applicable to more complex volume reaction situations.

Such a case of surface reaction on spheres is afforded by round colloidal particles bearing catalysts, and if the reaction rate be constant the present conditions apply. For reaction rates proportional to the concentration, boundary conditions $\partial c/\partial n=kc$ would be needed; which might as well be taken at once in the form $\partial c/\partial n=kc+b$, since it would be no more difficult and would be of more general use. For instance, the latter form will serve directly for volume reaction cells with high internal diffusion coefficient; for then the internal concentration is sensibly constant, say $=A$, and the flow through unit surface is $h(c-A)$ where $h=$membrane permeability. Equating this to the surface inflow

$$D\cdot\partial c/\partial n=h(c-A) \quad \text{or} \quad \partial c/\partial n=ch/D-hA/D$$

of the form above.

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Atmospheric Potential Gradient Anomalies

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A series of observations of the atmospheric potential gradient have been made over a period of six years and have been correlated with positions of the sun-spots and weather conditions. Maxima or minima of the potential gradient and storms appear to recur at intervals of approximately 27 days and at the same time that sun-spots appear near a plane through the axis of the sun and the earth. The supposition is made that the variations of the potential gradient and the storms are associated with radiation which travels approximately with the velocity of light from the sun. Since the light received from the sun-spot group cannot change rapidly as the group passes its position of nearest approach to the earth it seems more reasonable to suppose that a radiation not distributed according to Lambert's cosine law is involved.

PART I

ON undisturbed days the atmospheric potential gradient is positive, points higher above the surface of the earth being at higher potentials. The normal mean value at stations where records are kept is of the order of 100 volts per meter. The diurnal variation has important 12-hour and 24-hour components.[1]

During periods of disturbance the gradient rises or falls by very large amounts and may even reach high negative values. It is the times of occurrence of these disturbances that have principally engaged my attention during a long series of observations. For abbreviation in what follows a disturbance characterized by high positive values will be referred to as a maximum, one characterized by negative values as a minimum. It will be understood that the variation during disturbed conditions is so rapid and erratic that a "maximum" may in fact be a group of many true maxima with normal or subnormal minima between them. Frequently, however, a "maximum" has a continuously high value of more than an hour's duration, and a "minimum" may be equally prolonged.

Since the absolute values, normal or disturbed, were of no interest for my purpose, a very simple method of measurement was adequate. The potential, above ground, of a suitable electrode about 15 meters above the earth's surface was measured by a quadrant electrometer operated with lamp and scale at a sensitivity of about 10 volts per mm scale division. The electrode was the tip of a copper tube projecting 2 meters from an attic window in the Sloane Laboratory, which is on a hill in a part of New Haven conveniently distant from factory chimneys. The electrode was kept at the potential of the air near it by allowing water to flow from it at a rate of 2.5 cm^3 $sec.^{-1}$. (This rate is not at all critical.) The sulphur-insulated tank feeding this water dropper was in the observing room to keep it from freezing during cold weather.

Electrometer readings have been taken at 15-minute intervals, or more frequently, during a total of about 10 hours per day on about 270 days per year during the past seven years. The series began in November 1927 and was interrupted during July, August, and September of each year and during the period October 1932 to September 1933. During a small part of the time covered by the visual observations a second electrometer was used with a recording device to check the fact that no important disturbance was missed by the more usual procedure. An effort was made to distribute observations so as to detect the occurrence of maxima or minima on as many consecutive days as possible. This was the easier because a disturbed condition usually persists for more than an hour, so that absolutely continuous vigilance was unnecessary.

Fig. 1 represents the observations taken on March 14, 1928. The deflections of the electrometer, in centimeters, are plotted as ordinates and the time is plotted as abscissae. In the cases such as represented in Fig. 1, the disturbance was very large and the deflection of the electrometer went entirely off the scale. Whenever this happened, it is represented by a straight line with arrows, in the figure.

[1] H. W. Fisk and J. A. Fleming, Terr. Mag. **33**, 37 (1928).

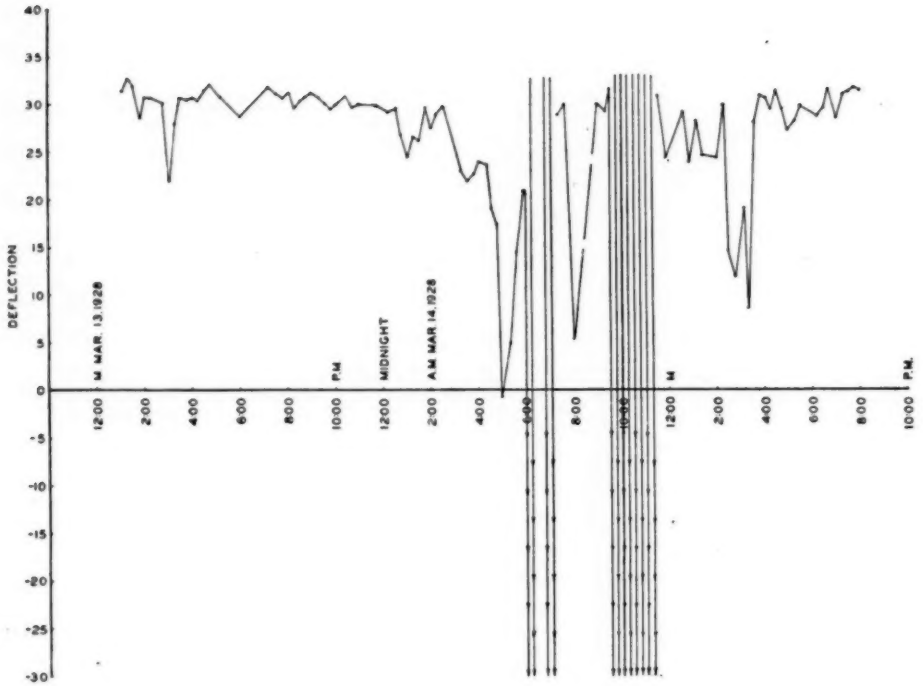

FIG. 1. Record of electrometer deflections for a single day.

An extensive series of such data can be used in a variety of ways and the first thing actually tested was the possibility of a correlation between the potential gradient and the weather. This is especially difficult to establish because there is so much variety in the weather at New Haven that one is very likely to find some abnormality of the weather available at about the time of any maximum or minimum. An apparent interdependence must then be reported with great caution. Subject to this proviso, then, it can be stated that throughout the years for which comparisons have been made, stormy weather is coincident with *minima* of the potential gradient, and that when only maxima occur the skies are generally clear. Days without maxima or minima are unlikely to be abnormal as to weather. Examples of all these cases can be found in Table I which gives a list of the disturbances in the potential gradient at New Haven for a period of one year beginning with October 1931, preceded by one sequence in 1927 which contains the example shown in Fig. 1. The list is confined to a period of one year for the sake of brevity. In the tables, the dates of the observed disturbances are given in the first column. The intervals in days between two successive disturbances (as between November 26, 1927 and December 23, 1927) are given in the second column. The nature of the potential gradient, whether positive or negative or both, is recorded in the third and fourth columns, while the condition of the weather is recorded in the last column. It will be seen from the table that the interval between disturbances is about 27 days.

It had been suggested by Bauer and Chree[2] that the potential gradient might show some dependence on sun-spots, like that found by Peters and Ennis[3] in the case of earth-current abnormalities, and that found by Bartels[4] in the case of magnetic storms. Table I shows that such a dependence actually exists as the intervals between potential disturbances, in any one sequence, approximate closely to the 27-day period of the sun's rotation. Every maximum and every minimum of the potential gradient can be assigned to a sequence of several members of the same kind following each other at inter-

[2] L. A. Bauer and C. Chree, Terr. Mag. **29**, 161 (1924).
[3] W. J. Peters and C. C. Ennis, Terr. Mag. **31**, 57 (1926).
[4] J. Bartels, Terr. Mag. **37**, 1 (1932).

TABLE I. *Record of observation on potential gradient.*

Date	Interval days	Gradient Positive	Gradient Negative	Weather conditions
Nov. 26, 1927	—	—	large	rain
Dec. 23	27	small	small	cloudy, snow
Jan. 19, 1928	27	—	large	rain, hail
Feb. 15	27	—	small	cloudy, foggy
Mar. 14	28	—	large	rain, foggy
Apr. 11	28	small	large	rain
Oct. 25, 1931	—	—	small	cloudy, thunder shower
Nov. 19	25	—	small	cloudy, sprinkle
Dec. 14	25	—	large	rain
Jan. 9, 1932	26	—	large	rain, snow
Feb. 4	26	small	large	rain, hail, snow
Feb. 29	25	small	—	cloudy, sprinkle
Mar. 26	26	large	small	cloudy, rain
Oct. 28, 1931	—	—	small	rain
Nov. 24	27	—	small, general*	foggy
Dec. 22	28	—	large	rain, fog
Jan. 19, 1932	28	large	—	fair
Feb. 16	28	small	—	fair
Mar. 14	27	large	small	snow, blizzard
Oct. 29, 1931	—	—	small	rain
Nov. 24	26	—	small, general	foggy
Dec. 20	26	—	small	rain, snow
Jan. 15, 1932	26	—	small	cloudy, foggy
Feb. 9, 10	26	two positives	large	snow, rain
Mar. 6, 7	26	large (two)	large	rain, hail
Oct. 14, 15, 16, 1931	—	—	small	rain some
Nov. 12	27	—	small	rain some
Dec. 9	27	—	large	rain
Jan. 6, 1932	28	—	large	rain, fog
Feb. 2	27	small	large	snow, rain
Feb. 29	27	small	—	cloudy, sprinkle
Mar. 27, 28	28	(two) small	large	rain
Oct. 28, 1931	—	—	small	little rain
Nov. 23	26	—	small, general	very foggy
Dec. 20	27	—	small	rain, little snow
Jan. 14, 1932	25	small	—	fair
Feb. 8	25	—	small	rain
Mar. 6, 7	27	two positives	large	rain, hail
Apr. 2	27	small	small	cloudy, sprinkle
Nov. 5, 1931	—	—	small, very	fair
Dec. 4	28	—	large	rain
Jan. 1, 2, 1932	27	—	large	rain, snow
Jan. 29, 30	28	small	(two) negatives	rain, foggy
Feb. 24, 25, 26	27	(three) large	(one) negative	snow
Mar. 23, 24, 25	28	(three) large	small	rain
Oct. 25, 1931	—	—	small	thunder shower
Nov. 21	27	—	small	very foggy
Dec. 18	26	small	—	fair
Jan. 14, 1932	27	large	—	fair
Feb. 11, 12	28	(two) large	small	rain, very foggy
Mar. 10	28	small	—	fair
Nov. 1, 1931	—	—	small	fair
Nov. 27	26	—	large	snow
Dec. 25	28	—	small	fair
Jan. 21, 1932	27	small	small	snow, rain
Feb. 16, 17	27	large	large	snow, rain
Mar. 14	27	large	large	snow

* General means that the potential gradient was disturbed more or less for a considerable time.

TABLE I—(*Continued*).

Date	Interval days	Gradient Positive	Gradient Negative	Weather conditions
Oct. 28, 29, 30, 1931	—	—	(three) small	rain
Nov. 22, 23, 24	25	—	(three) small	cloudy, foggy
Dec. 17, 18	25	(two) small	—	fair
Jan. 11, 12, 1932	25	(two) small	—	rain, foggy
Feb. 6, 7	26	(two) large	—	fair
Mar. 3, 4	26	(two) small	—	fair
Mar. 30, 31	27	(two) large	small	rain
Nov. 7, 1931	—	small	—	fair
Dec. 3	26	small	—	fair
Dec. 28	25	small	—	fair
Jan. 22, 23, 1932	26	—	(two) negatives	cloudy, sprinkle
Feb. 18, 19	27	(two) small	(one) small	fair
Mar. 16, 17	27	(two) large	(two) large	rain, snow
Nov. 7, 1931	—	small	—	fair
Dec. 4	27	—	large	rain
Dec. 31	27	small	—	fair
Jan. 26, 27, 1932	26	—	(two) large	rain
Feb. 21, 22	26	(two) large	—	fair
Mar. 18, 19	26	(two) large	large	rain, snow, hail
Nov. 3, 1931	—	—	small	sprinkle, cloudy
Nov. 30	27	—	large	rain
Dec. 28	28	small	—	fair
Jan. 25, 1932	28	small	—	fair
Feb. 21	27	large	—	fair
Mar. 21	29	large	large	cloudy, snow
Nov. 15, 1931	—	—	small	rain, some
Dec. 11	26	—	small, general	rain, foggy
Jan. 6, 1932	26	—	large	rain
Feb. 2	27	small	large	snow, rain
Feb. 29	27	small	small	cloudy, sprinkle
Mar. 27, 28	27	(two) small	large	rain

vals, on the average, of 27 days. Naturally, these sequences overlap one another as regards time. Such sequences were found among the disturbances during all of the six years studied. An analysis of weather conditions during the above period shows that unusual weather conditions can also be assigned to sequences of several members following each other at intervals of approximately 27 days.

These two sets of sequences include every maximum and every minimum of the potential gradient and also every storm (either rain or snow) that occurred during the entire six years studied. The number of such groups or sequences for each of the six years studied is given below.

1927–1928	12 groups	1930–1931	12 groups
1928–1929	12 "	1931–1932	13 "
1929–1930	14 "	1933–1934	13 "

Where a series of maxima or minima seems to be complete the intensity or duration, or both, increase to a maximum and then decrease again, so that we are led to look for a feature of the sun's surface which grows and decays during a total time of five or six rotations of the sun (with respect to the earth). Sun-spots and especially sun-spot groups, are such features.

While the whole period of observation is less than a single sun-spot cycle, there is some evidence that a long period change in the nature of potential gradient "storms" is under way. In the year 1927–28 minima were frequent and intense. Since 1930–31 maxima have predominated.

For one period of six months—November 1927 to May 1928—in which many disturbances were observed and followed, a comparison has been made with solar photographs furnished through the courtesy of the U. S. Naval Observatory. The series of photographs covered every clear day at the Observatory, and shows in many cases groups of spots too small to be listed in computing the sun-spot number. The time of

TABLE II. *Comparison of time of transit of sun-spots across the plane through the axis of the sun and earth and disturbances of potential gradient at various locations.*

Date, 1928	Photo time	Max. time Australia	Max. time Peru	Max. time Wash., D. C.	Max. time New Haven	Date, 1928	Photo time	Max. time Australia	Max. time Peru	Max. time Wash., D. C.	Max. time New Haven
Mar. 1	34	?	—	—	34	Mar. 11	1	?	?	4	—
1	66	66	—	?	65	11	6	?	?	6	—
1	70	67	—	73	73	11	12	12	?	10	—
2	19	19	?	19	—	11	38	?	?	36	36
2	20	20	?	20	—	11	48	52	?	52	52
2	50	48	49	?	50	11	72	72	?	72	72
3	13	—	12	18	—	11	78	78	?	78	78
3	21	—	19	19	—	11	92	92	?	92	92
3	23	—	20	20	—	12	11	?	?	10	11
3	27	29	?	26	—	12	14	?	14	14	14
3	31	?	31	30	—	12	25	?	26	26	26
3	38	?	38	38	—	12	45	47	?	46	?
3	45	44	?	?	44	12	66	?	?	66	?
3	57	57	57	57	57	12	72	75	?	75	—
3	65	—	64	—	64	12	92	?	92	92	—
3	67	?	68	?	69	13	23	28	27	27	—
3	90	90	—	?	90	13	32	32	32	32	32
3	93	92	—	?	?	13	60	?	?	59	60
4	1	4	4	1	—	13	66	?	?	68	68
4	28	26	—	26	—	13	76	76	76	76	—
4	49	49	—	49	49	14	4	?	6	4	4
4	63	64	—	?	64	14	12	?	?	12	12
4	67	?	—	67	69	14	19	?	?	19	19
4	74	70	70	70	70	14	20	23	?	20	20
4	80	80	81	?	81	14	28	?	?	28	28
4	92	92	92	92	—	14	39	39	39	39	39
5	10	?	8	?	—	14	41	41	41	41	41
5	20	21	21	21	—	14	60	60	60	?	60
5	34	?	35	35	—	14	79	80	80	?	?
5	37	?	36	38	39	15	26	26	26	26	—
5	74	74	74	75	74	15	30	32	32	32	—
5	76	80	76	76	76	15	48	54	54	54	—
5	94	?	92	94	94	15	66	68	68	?	68
6	10	?	6	?	6	15	81	?	82	82	—
6	16	?	14	16	?	16	10	12	12	12	10
6	38	38	38	38	?	16	15	15	15	15	—
6	65	?	65	65	—	16	19	?	19	19	?
6	92	91	91	92	91	16	28	28	?	?	?
7	2	?	4	4	4	16	32	32	32	32	—
7	20	20	20	20	20	16	39	39	?	39	—
7	25	24	24	24	24	16	44	44	44	44	—
7	29	32	?	32	—	16	73	?	73	73	73
7	37	36	?	38	38	16	76	76	76	76	—
7	43	42	?	42	42	17	48	48	?	48	48
7	48	48	49	?	—	17	60	?	?	58	58
7	58	?	?	59	59	17	69	68	69	?	69
7	63	62	62	62	62	17	89	?	?	89	—
7	69	?	?	68	70	18	4	?	4	4	—
7	76	?	76	—	?	18	12	12	12	12	—
7	90	90	?	?	—	18	17	16	16	16	—
8	42	42	42	?	42	18	51	?	53	—	53
8	85	85	?	85	—	18	55	?	56	—	56
8	87	87	87	87	—	18	60	?	60	—	60
8	96	?	96	96	—	18	69	?	69	—	69
9	42	?	42	42	42	18	82	?	82	—	82
9	44	?	44	44	44	18	92	?	92	—	—
9	64	64	?	64	64	19	53	52	?	—	?
9	72	70	?	71	68	19	56	56	?	—	56
9	78	80	?	80	80	19	58	60	60	—	60
9	82	84	?	82	82	19	67	?	68	—	68
9	96	96	?	96	96	19	71	?	71	—	71
10	8	8	?	8	8	19	80	?	76	—	—
10	14	14	?	14	14	19	92	?	92	—	—
10	22	22	?	?	22	20	54	54	54	—	54
10	36	36	36	36	—	20	82	82	82	—	82
10	62	61	61	?	61	21	38	38	?	—	38

TABLE II.—(*Continued*).

Date, 1928	Photo time	Max. time Australia	Max. time Peru	Max. time Wash., D. C.	Max. time New Haven	Date, 1928	Photo time	Max. time Australia	Max. time Peru	Max. time Wash., D. C.	Max. time New Haven
Mar. 21	47	47	46	—	47	Mar. 27	36	?	36	36	36
21	62	60	60	?	—	27	40	?	40	40	40
21	76	76	76	—	—	27	45	48	47	?	46
22	20	21	21	—	—	27	49	50	48	50	—
22	55	54	54	?	54	27	72	76	76	80	76
22	90	90	90	90	90	28	18	20	?	?	—
23	20	20	20	?	—	28	20	24	28	28	—
23	54	56	56	?	—	28	38	?	38	?	—
23	56	60	60	?	—	28	43	?	43	43	—
23	62	62	62	?	62	28	76	?	?	80	80
23	76	78	78	78	—	29	18	20	?	20	—
24	28	?	28	26	—	29	23	24	?	22	—
24	51	?	51	?	—	29	88	88	88	88	88
24	56	?	56	54	—	29	91	91	91	91	91
24	63	?	64	64	—	29	93	?	?	92	92
24	68	?	68	?	—	30	7	?	?	7	7
25	10	14	14	14	—	30	10	10	10	10	10
25	65	?	66	66	66	30	14	16	16	16	16
26	6	6	?	?	—	30	31	?	?	32	32
26	12	12	?	?	—	30	40	?	?	40	40
26	16	16	16	16	—	30	54	?	?	54	54
26	25	?	?	28	28	30	58	58	58	58	58
26	34	?	34	34	34	30	67	67	67	67	67
26	38	?	38	38	38	30	69	68	68	68	68
26	43	?	44	44	44	30	76	?	?	76	—
26	81	?	?	80	—	30	88	86	86	?	86

passage of each spot shown on these photographs through a plane determined by the axis of the sun and the earth can, of course, be computed. The time of a large disturbance of potential gradient, maximum or minimum, always agrees with the occurrence of one or more small spots near the critical plane just defined. The detailed data giving the times of passage of spots through this plane for March 1928 and the times of disturbances of the potential gradient during this period will be given in Part II, a second paper to follow.

Comparison of the whole series of solar photographs with the electrometer records suggests that large quiescent spots are not as effective in disturbing the potential gradient as small but rapidly changing spots. Parallel findings by Abbot[5] in regard to the solar radiation suggest that the atmospheric potential gradient is closely related to radiation from the sun. Since the light received from a sun-spot group cannot change rapidly as the group passes its position of nearest approach to the earth, it seems reasonable to suppose that a radiation not distributed according to Lambert's cosine law is here involved.

[5] C. G. Abbot, Proc. Nat. Acad. Sci. **9**, 355 (1923).

PART II

In Part I of this paper is given a series of observations covering a period of six years showing the times of extreme disturbance of the earth's potential gradient and it was stated that these times agreee with the times of passage of sun-spot groups across a plane through the sun's axis and the earth. Part II compares the actual calculated times of transit of the sun-spots with the times of extreme disturbance of the potential gradient as observed by me in New Haven and by other observers at Washington, Peru and Australia. The comparison has been made by aid of solar photographs furnished through the courtesy of the U. S. Naval Observatory. This series of photographs covered every clear day at the Observatory, and shows, in many cases, groups of spots too small to be listed in computing the sun-spot number. The time of passage of each spot shown on these photographs through a plane determined by the axis of the sun and the earth was computed for each day of the month of March, 1928. The time of a large disturbance of potential gradient, maximum or minimum, always agrees with the occurrence of one or more small spots near the

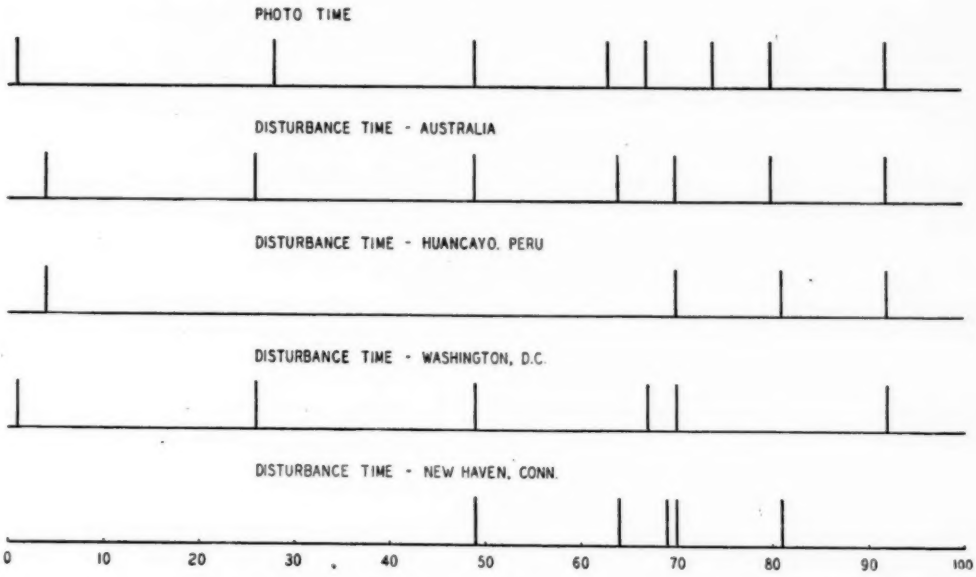

FIG. 2. Comparison of time of transit of sun-spots across the plane through the axis of the sun and earth and disturbances of potential gradient at various locations.

critical plane just defined. The times of passage of spots through this plane are given in the second column of Table II. The month of March 1928 was chosen because, during this month, the potential gradient was much disturbed and also there were a large number of sun-spots at this time. A visual comparison of the Naval Observatory photographs for a period covering November 1927 to May 1928, with my observations on the potential gradient disturbances at New Haven during the same period, appeared completely in agreement with this accurate study for March 1928. It will be understood that the variation of the potential gradient during disturbed conditions is so rapid and erratic that a "maximum" may in fact be a group of many true maxima with normal or subnormal minima between them.

In the first column of Table II are given the days of the month. The calculated times of transit of sun-spots across the plane through the axis of the sun and the earth are given in the second column. The times of the disturbances of the potential gradient at Watheroo, Australia; Huancayo, Peru; Washington, D. C.; and New Haven, Connecticut, are given in the third, fourth, fifth, and sixth columns, respectively. These stations all lie nearly on the same meridian.[6] All of the times are recorded in quarter hours for a 24-hour day. For example, a disturbance of the potential gradient that occurred at New Haven, Connecticut, at 4 P.M. on March 3, 1928, is recorded as the 64th quarter-hour of that day and a disturbance that occurred at 5 A.M. on March 7, 1928, is recorded as the 20th quarter-hour of March 7th. A dash signifies that no observations were taken on that day and an interrogation point means that, if there were any disturbance of the potential gradient, the effect was small. The photostat copies of the potential gradient records made at Watheroo, Australia; Huancayo, Peru; and Washington, D. C. were obtained through the courtesy of the Smithsonian Institution. These data are plotted one above the other (in the order given above) together with the records made at New Haven, Connecticut, all being reduced to Washington time. Fig. 2 is a graphical representation for the 24-hour day of March 4, 1928. The calculated times of transit of sun-spots across the critical plane through the axis of the sun and the earth are compared with the times of disturbance of the potential gradient at Watheroo, Australia;

[6]

	Latitude	Longitude	
Watheroo, Australia	30° 18′.9 S	115° 52′.6	E from
Huancayo, Peru	12° 02′.7 S	284° 40′	E Green-
Washington, D. C.	38° 57′.4 N	282° 56′	E wich
New Haven, Connecticut	41° 18′.9 N	72° 57′	W

TABLE III. *Correlation of duration of disturbance of potential gradient and time of passage of a group of sun-spots through a given point.*

Date	Calculated time for group to pass a single point	Observed time of duration of disturbance of potential gradient
March 6, 1928	16.0 hours	14+hours
7	25.2	26
8 and 9	23.4	24
14	11.5	12
15 and 16	21.6	22
17 and 18	18.0	18+
24	21.6	24
26	5.5	6
29 and 30	21.6	22

Huancayo, Peru; Washington, D. C.; and New Haven, Connecticut as given in Table II.

When these data are plotted as stated above it is found that, in a few cases, the maximum total effect in the potential gradient is displaced in time as one passes in turn from Watheroo, Australia, to Huancayo, Peru; Washington, D. C.; and New Haven, Connecticut. This maximum effect appears first in Australia and lastly at New Haven. Moreover, in almost every instance the potential gradient at Watheroo, Australia, shows a positive maximum but as one passes, in turn, to Huancayo, Peru, Washington, D. C. and New Haven, Connecticut, the positive maximum is also accompanied by a negative minimum of the potential gradient which increases in magnitude as one approaches New Haven. The negative minimum is also always accompanied by rain or snow.

The largest of these displacements of the potential gradient, maximum or minimum, occurred on the days of March 9, March 11, March 14, March 18, and March 30, 1928. For these days there were also large groups of sun spots on the photographs of the sun. The time required for these entire groups of spots to pass a given point has been calculated for each of the above days and these times compared with the duration of the potential gradient disturbance as shown by the records. These results are given in Table III. It will be observed that the time necessary for any group to pass a given point agrees very closely with the time of duration of the potential gradient disturbance.

Because of the displacements of the total maximum effects mentioned above, some of the disturbances of the potential gradient (recorded in Table II) that occurred in Australia, for instance, at the same time that a disturbance occurred at New Haven, Connecticut, may not be associated with the same individual sun-spot. The fact to be observed, however, is that the time of a large disturbance of potential gradient, maximum or minimum, always agrees with the occurrence of one or more small spots near the critical plane mentioned above, although the disturbance of the potential gradient at some stations was sometimes small while it was large at other stations. The list given in Table II includes every large disturbance of the potential gradient for the month of March 1928.

The author desires to express his very great appreciation to the U. S. Naval Observatory for their kindness in furnishing the photographs of the sun, also to the Smithsonian Institution for furnishing the photostat copies of the potential gradient records.

The Motion of a Coil Spring Having Variable Tension and Variable Mass Per Unit Length

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The problem here treated is that of finding the longitudinal motion of a long coil spring suspended vertically from a rigid support: the spring being sufficiently "soft" and long that it stretches appreciably under its own weight. Thus the tension and linear density each vary from top to bottom of the spring. A general solution involving displacements from the stretched static position is obtained and applied to three specific sets of boundary conditions; first, the lower end free, second the lower end rigidly fixed and lastly with a weight attached to the lower end. The natural periods corresponding to the first three fundamental modes of motion for each of the three conditions mentioned above have been determined experimentally for a long coil spring of piano wire and the results compared with the theoretically computed values. Before experimental results are given it is shown, by a second form of the equation of motion, that the natural frequencies of a spring are not affected by stretching.

INTRODUCTION

A SEARCH through the literature indicates that a solution to the problem outlined in the abstract has not been given. The motion of suspended springs, cables and rods has been determined[1] assuming that the mass per unit length is not appreciably changed by their own weight or even by an attached weight. However, this may be far from true for a long cable or coil spring.

Elongations involving Hooke's law are usually considered from the unstrained position. However, for this particular problem, it is of more interest to know displacements from the stretched equilibrium position since for a soft spring the stretched length may be several times that of the unstretched length. Thus in the treatment which follows displacements due to the motion of the spring are considered *relative to the stretched static position*. Likewise the distance to any point is measured *along the stretched equilibrium position*.

THE DIFFERENTIAL EQUATION OF MOTION

It is evident that the total displacement, E_1, of any point on the spring which is acted on by gravity plus forces due to wave motion, is given by

$$E_1 = E + E_0, \tag{1}$$

where E_0 represents the static displacement due to gravity and E the displacement from the static position due to wave motion. It is also evident that the distance x (see Fig. 1) to a point on the spring, measured along the stretched equilibrium position, is given by

$$x = x_0 + E_0, \tag{2}$$

where E_0 has the meaning given above and x_0 is the position of the same point as measured along the unstretched spring.

Now the net force F on an element of spring dx, in terms of the total tension in the spring at a point x due to gravity and wave motion, is given by

$$F = (\partial T/\partial x)dx.$$

But from Eqs. (1) and (2) and the fact that by Hooke's law $\partial E_0/\partial x_0 = T_1/k$, where T_1 is the tension due to gravity alone and k the usual spring constant, it follows that[2]

$$T = (k + T_1)(\partial E/\partial x).$$

Hence, $F = (\partial/\partial X)[(k+T_1)(\partial E/\partial x)]dx.$

To obtain the differential equation of motion we set F equal to the mass times the acceleration of the element dx. That is,

$$(\partial/\partial x)[(k+T_1)(\partial E/\partial x)] = m\partial^2 E/\partial t^2, \tag{3}$$

where m is the variable mass per unit length of the spring. Elementary considerations show that m is given in terms of m_0, the linear density of the unstretched spring, by

$$m = km_0/(k+T_1).$$

[1] S. Timoshenko, *Vibrations Problems in Engineering* (D. Van Nostrand Publishing Co.).

[2] It is to be noted that this amounts to a modified form of Hooke's law which is applicable when it is desired to measure distances along a spring that is initially stretched and displacements relative to the initially stretched condition. It seems that this form, which is usually omitted from intermediate texts, should at least be mentioned.

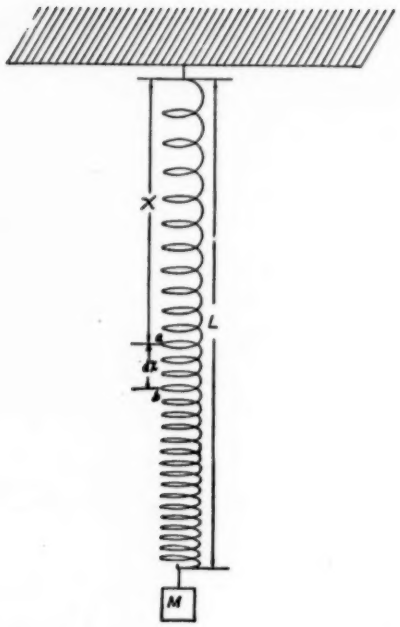

FIG. 1. Showing arrangement of spring and notation used.

Therefore (3) becomes

$$(\partial/\partial x)[(k+T_1)\partial E/\partial x] = (km_0/(k+T_1))\partial^2 E/\partial t^2. \quad (4)$$

T_1 is a function of x which can easily be found as follows:

$$dT_1 = -mgdx = -(km_0/(k+T_1))dx.$$

Thus, $$kT_1+T_1^2/2 = -km_0gx+c.$$

Evaluating the constant c by putting $T_1=Mg$ at $x=L$, where M is an attached mass and L the stretched static length of the spring, we get

$$T_1+k=[(k+Mg)^2+2km_0g(L-x)]^{\frac{1}{2}}.$$

Eq. (4) can now be greatly simplified by making the substitution: $y=T_1+k$. That is,

$$y^2=(k+Mg)^2+2km_0g(L-x) \quad (5)$$

since by (5) $\partial y/\partial x=-km_0g/y$, it follows that $\partial/\partial x=-(km_0g/y)\partial/\partial y$. Eq. (4) can thus be written:

$$-\frac{km_0g}{y}\frac{\partial}{\partial y}\left[y\left(\frac{-km_0g}{y}\right)\frac{\partial E}{\partial y}\right]=\frac{km_0}{y}\frac{\partial^2 E}{\partial t^2}.$$

Finally we get:

$$\partial^2 E/\partial t^2 = km_0g^2(\partial^2 E/\partial y^2). \quad (6)$$

It should be remembered that E of this equation is measured from the stretched equilibrium position when gravity alone is acting.

The general solution of this well-known form of the wave equation is

$$E=f_1(y+\alpha t)+f_2(y-\alpha t),$$

from which particular solutions to fit any given boundary conditions can be found. The three sets of boundary conditions mentioned in the abstract will now be considered one at a time.

Case one. $M=0$ and the lower end free.

The boundary conditions for this case are as follows:

(1) Assume that the motion starts from rest, which means that at $t=0$, $\partial E/\partial t=0$.

(2) At $x=0$, $E=0$ at all times or in terms of the y variable; at $x=0$, $y=[(k+Mg)^2+2km_0gL]^{\frac{1}{2}}=y_0$. Therefore this condition becomes, at $y=y_0$, $E=0$.

(3) When $t=0$, $E=f(x)$ or when $t=0$, $E=\varphi(y)$.

(4) At $x=L$, $F=(k+T_L)(\partial E/\partial x)=0$ and since $T_L=0$, $\partial E/\partial x=0$.

Or again, at $x=L$, take $y=y_L$. Then it easily follows that at $y=y_L$, $\partial E/\partial y=0$.

In order to satisfy these boundary conditions choose the following particular solution of the differential equation of motion.

$$E=D\cos\beta t\sin\beta/\alpha(y_0-y).$$

It is seen that this solution satisfies conditions (1) and (2) and will satisfy (4) if $\beta/\alpha(y_0-y_L)=\pi(n+\frac{1}{2})$ where $n=0, 1, 2, \cdots$. In order to satisfy condition (3) we see that at $t=0$,

$$E=D\sin\frac{\pi(n+\frac{1}{2})}{y_0-y_L}(y_0-y)$$

and therefore in order that $E=\varphi(y)$ at $t=0$ we must develop $\varphi(y)$ into a Fourier series of type

$$\varphi(y)=\sum_{n=0}^{\infty} D_n\sin\frac{\pi(n+\frac{1}{2})}{y_0-y_L}(y_0-y).$$

This can easily be done for any given initial distortion. Thus the final solution becomes:

$$E=\sum_{n=0}^{\infty} D_n\cos\frac{\pi\alpha(n+\frac{1}{2})}{y_0-y_L}t\sin\frac{\pi(n+\frac{1}{2})}{y_0-y_L}(y_0-y).$$

It is seen that β determines the fundamental frequencies of the spring.

$$\beta = 2\pi\nu_n = \pi\alpha(m+\tfrac{1}{2})/(y_0-y_L),$$

where ν_n is any one of the fundamental frequencies of motion. From this it follows that the periods are given by:

$$p_n = \frac{2(y_0-y_L)}{\alpha(n+\frac{1}{2})} \text{ which reduces to } p_n = \frac{4l_0(m_0/k)^{\frac{1}{2}}}{2n+1},$$

where l_0 is the unstretched length of the spring.

Case two. Lower end rigidly fixed at the point where the spring comes to rest when free.

The method of handling this case is identically the same as that of case one. When we take into account appropriate boundary conditions, the following solution is obtained:

$$E = \sum_{n=0}^{\infty} C_n \cos\frac{n\pi t}{y_0-y_L} \sin\frac{n\pi}{y_0-y_L}(y_0-y).$$

The fundamental periods are given by

$$p_n = \frac{2(y_0-y_L)}{n\alpha} = \frac{2l_0(m_0/k)^{\frac{1}{2}}}{n}.$$

Case three. Mass M attached to the lower end.

This case is similar to the first two except that the boundary conditions having to do with the lower end, and the Fourier development are a little more difficult to handle. The condition to be satisfied at the lower end (where $x=L$) is that the force F producing motion must at all times be given by $F=-M(\partial^2E/\partial t^2)=(k+T_1)(\partial E/\partial x)$, but $T_1=Mg$ at $x=L$; thus

$$(k+Mg)(\partial E/\partial x) = -M(\partial^2E/\partial t^2).$$

In terms of the y variable this becomes

$$2km_0g((k+Mg)/2My_L)(\partial E/\partial y) = \partial^2E/\partial t^2.$$

The particular solution $E = Q\cos\beta t \sin\beta/\alpha(y_0-y)$ will satisfy this boundary condition provided β satisfies the relation

$$2km_0g((k+Mg)/2My_L\alpha\beta) = \tan\beta/\alpha(y_0-y_L)$$

obtained by substituting the above solution into the boundary condition. Roots of this equation with β considered as a variable can easily be obtained graphically. It is thus seen that $E=\sum Q_n \cos\beta_n t \sin\beta_n/\alpha(y_0-y)$ where $\beta_1, \beta_2, \beta_3, \cdots$ are consecutive roots of the above tangent equation, will satisfy all boundary conditions for any given initial distortion $\varphi(y)$. In this case the method of obtaining the values of Q_n is somewhat different from that of the ordinary Fourier coefficients.[3] It is obvious that the fundamental periods of motion are given by

$$p_n = 2\pi/\beta_n.$$

Another Form of the Equation of Motion

Another form of the equation of motion which is of interest, particularly since it shows that the natural frequencies of motion are exactly the same for a stretched as for an unstretched spring, is easily obtained as follows.

By Eq. (1), since E_0 is constant in time, it follows that

$$\partial^2E_1/\partial t^2 = \partial^2E/\partial t^2.$$

Again we write the net force on an element of length of spring dx as

$$F = (\partial T/\partial x)dx.$$

Therefore the equation of motion for the stretched spring can be put in the form

$$\partial T/\partial x = (km_0/k+T_1)(\partial^2E_1/\partial t^2), \qquad (7)$$

where each symbol has the original meaning given it. From Eq. (2) and the fact that, by Hooke's law,

$$T_1 = k\partial E_0/\partial x_0$$

it follows that

$$\partial/\partial x = (k/(k+T_1))\partial/\partial x_0. \qquad (8)$$

Also by Hooke's law,

$$T = k\partial E_1/\partial x_0. \qquad (9)$$

Making use of (8) and (9), Eq. (7) can be written

$$\frac{k}{k+T_1}\frac{\partial}{\partial x_0}\left(k\frac{\partial E_1}{\partial x_0}\right) = \frac{km_0}{k+T_1}\frac{\partial^2E_1}{\partial t^2}$$

or finally,

$$(k/m_0)(\partial^2E_1/\partial x_0^2) = \partial^2E_1/\partial t^2. \qquad (10)$$

It is seen that this is exactly the equation of motion of an unstretched spring. Hence the natural frequencies of motion are unaffected by stretching whether by gravity or otherwise, for the derivation of (10) does not depend on a

[3] W. E. Byerly, *Fourier's Series and Spherical Harmonics*, pp. 119–121.

TABLE I. *Comparison of computed and measured periods of a spring. Case I, spring hanging and lower end free. Case II, lower end fixed at point where spring rests when free. Case III, mass attached to lower end.*

Period	Case I		Case II		Case III	
	Computed	Experimental	Computed	Experimental	Computed	Experimental
P_0	2.263	2.262				
P_1	0.754	0.752	1.131	1.123	2.64	2.64
P_2	.452	.451	0.565	0.564	0.863	0.866
P_3	.323	.323	.371	.374	.51	.51

particular kind of initial static distortion. This means for example, that the natural frequencies of motion of a spring, either end of which is rigidly fastened to a support, is independent (within certain limits of course) of the distance between supports. However the wave-length of a particular mode of motion will depend on the distance between supports. Likewise it is evident that the wave-length of a particular mode of motion in a vertically suspended soft spring will vary from top to bottom of the spring.

EXPERIMENTAL RESULTS

The natural periods corresponding to the first three fundamental modes of motion for each of the three cases discussed above have been determined experimentally for a long coil spring of piano wire and the results compared with theoretically computed values. With a little practice, it was found easy to excite manually any one of the fundamental modes of motion almost to the extinction of all others. When this was done, the period was found by timing with a stopwatch a hundred or more complete oscillations and repeating several times as a check. The experimental values listed are the averages of a large number of oscillations.

Data on the spring are as follows: diameter of coil 2.3 cm, diameter of piano wire 1 mm, unstretched length of coil $l_0=120.5$ cm, total mass of the spring 398.5 grams, equilibrium length suspended from rigid support with no weight attached $L=275.5$ cm, spring constant $k=1.5\times10^5$ dynes, $M=67.5$ grams (for case 3). Computed and experimental results are shown in Table I.

It is seen from Table I that the experimental results amply confirm the mathematical computations.

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